EXAM ONE

CS 340 Brooks
Spring 2009 NAME KEY – DO NOT DISTRIBUTE

This is an open book and closed notes exam.

1. (8 points) Use **complete sentences** to write a proof of the following statement about integers. **Use only the definitions odd and even, together with algebra**. Do not use any other known information about odd and even numbers.

If $5x + y$ is odd and x is even, then y is odd.

5x + y is odd, so 5x + y = 2j + 1 for some j $\in$ **Z**

x is even, so x = 2k for some k $\in$ **Z**

10k + y = 2j + 1 (algebra)

y = 2j – 10k + 1 (algebra)

y = 2(j – 5k) + 1 (algebra)

j – 5k is an integer, so y is in form of 2n + 1, therefore y is odd. QED.

2. (12 points) Evaluate each expression.

a. power($\{a, b\}$) = { Ø, {a}, {b}, {a,b} }

b. $\{a, b, c\} - (\{a, b\} \cap \{b, c, d\})$ = {a, b, c} – {b} = {a, c}

c. $\{a, b\} \times \{7, 8, 9\}$ = {(a,7),(a,8),(a,9),(b,7),(b,8),(b,9)}

d. floor($\log_2(34)$) = $\log_2 32 = 5$, $\log_2 64 = 6$, so floor($\log_2$ (34))=5

For the following expressions, let $f : \mathbf{N}_9 \rightarrow \mathbf{N}_9$ be defined by $f(x) = 3x \bmod 9$.

e. $f^{-1}(\{0, 4\})$ = Not a function

f. range(f) = {0, 3, 6}

3. (6 points) Let $A = \{3n + 6 \mid n \in \mathbf{N}\}$ and let $B = \{3k + 3 \mid k \in \mathbf{N}\}$. Prove that $A \subset B$.

A ∩ B = A then $A \subset B$

See if (3n + 6) can be expressed in the same form as B: 3k + 3

3n + 6 = 3(n+1) + 3. If $k \in \mathbf{N}$, then so is k + 1. Therefore $A \subset B$. *QED*.

4. (5 points) Find an expression for the number of strings over the alphabet $\{a, b, c, d\}$ that have length 6 and such that the first letter in each string is either a or b, and in which each string contains at least one c.

U = strings of length 6 that start with a or b. $|U| = 2 \cdot 4^5 = 2048$

T = strings of length 6 starting with a/b that do not contain "c". $|T| = 2 \cdot 3^5 = 486$

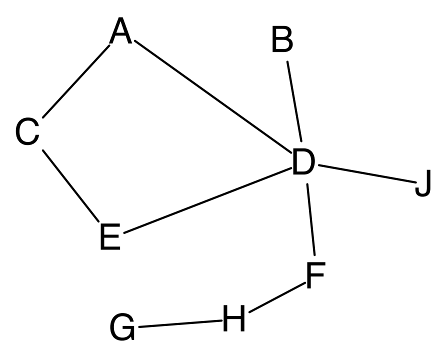
$|U - T| = |U| - |T| = 2048 - 486 = 1562$

5. (5 points) Solve the language equation for L.

$$\{\Lambda, b, ab\}L = \{\Lambda, a, b, ba, ab, aba\}.$$

$L = \{\Lambda, a\}$

6. (6 points) Given the following graph.



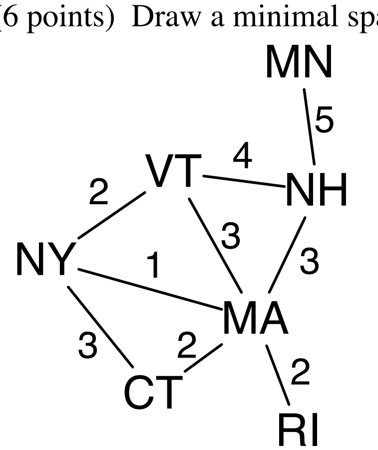
a. Write down the vertices of the graph in the order that they are visited by a breadth-first search of the graph that starts at vertex **D**.

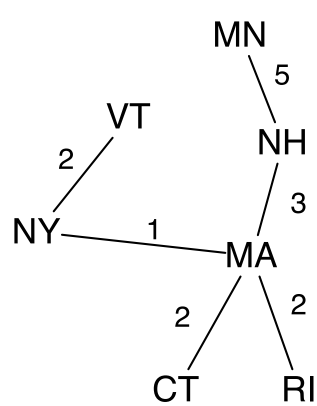
One solution: D, B, J, F, A, E, C, H, G

b. Write down the vertices of the graph in the order that they are visited by a depth-first search of the graph that starts at vertex **D**.

One solution: D, F, H, G, E, C, A, B, J

7. (6 points) Draw a minimal spanning tree for the following weighted graph.





8. (8 points) Let $\mathbf{Q}^+$ denote the positive rational numbers. Let $f : \mathbf{Q}^+ \to \mathbf{Q}^+$ be the function defined by $f(x) = x/(x + 3)$.

a. Show that f is injective.

Injective if f(x) = f(y) → x = y. y/(y+3) = x/(x+3). xy + 3y = xy + 3x. 3y = 3x. x = y. QED.

b. Show that f is NOT surjective.

BWOC, Let f(x) = 1. 1 = x/(x+3). x + 3 = x. 3 = 0. Therefore 1 is not reachable by f(x), so not surjective. QED.

9. (6 points) Let $f : \mathbf{N}_8 \to \mathbf{N}_8$ be defined by $f(x) = (3x + 1) \bmod 8$. Find a formula for f^{-1}, the inverse of f

gcd(3, 8) = 1 so there is an inverse. 1 = 3k + 8m for some m. Let m = -1, so 3k = 9, k = 3.
f(c) = 0 → c = 5.
f^{-1} = (kx + c) mod 8, so f^{-1} = (3x + 5) mod 8.

10. (6 points) Let S = {January, February, March, April, May, June, July} and suppose that h: $S \to \mathrm{N}_8$ is the hash function defined as follows, where | x | denotes the length of string x.

$$h(x) = |\, x \,| \bmod 7.$$

Use h to place each month of S into the following hash table starting with January, then February, and so on until July. Resolve collisions by linear probing with a gap of 2.

0	January
1	February
2	April
3	May
4	June
5	March
6	July

11. (4 points) Answer the following:

a. Find the cardinality of {2, 5, 8, … 35, 38}

Can also say f(x) = 2 + 3x, x 0…12, Therefore |S| = 13

b. Show that the set of negative integers is countable by establishing a bijection between the set an **N**.

Can express set of negative integers as f(x) = -x, $x \in \mathbf{N} - 0$
f(x) = f(y) → x = y is true, so injective.
Each $x \in \mathbf{N}^-$ can be expressed as f(x) = -x, so surjective. Therefore bijective and countable.
QED.

12. (12 points) Write out an inductive definition for each set.

a. $S = \{x \mid x \in \mathbf{Z} \text{ and } x \bmod 4 = 0\}$.

Ø ∈ **S** (basis)

if n ∈ **S** then n + 4 ∈ **S**

b. $S = \{x \mid x \in \text{Lists}(\{0, 1\}) \text{ and } x \text{ has even length}\}$.

$\langle\rangle \in$ **S** (basis)

if L ∈ **S** then for each pair (x,y) ∈ **{0, 1}**, x :: y :: L ∈ **S**

c. $S = \mathbf{N} \times \{a, b\}^*$.

$(0, \langle\rangle) \in$ **S** (basis)

if (x, y) ∈ **S** then (x+1, y) ∈ **S**

if (x, y) ∈ **S** then (x, a :: y) ∈ **S** and (x, b :: y) ∈ **S**

13. (16 points) Write a recursive definition for each of the following functions. (Write your definitions in either if-then-else form or as equations.)

a. $f : \mathbf{N} \to \mathbf{N}$ defined by $f(n) = 0 + 4 + 8 + \ldots + 4n$.

f(0) = 0

f(n + 1) = f(n) + 4(n+1)

b. $g : \text{Lists}(\mathbf{N}) \to \mathbf{N}$ defined by $g(\langle x_1, ..., x_n\rangle) = x_1 + \ldots + x_n$.

$g(\langle\rangle) = 0$

g(x :: L) = x + g(L)

c. $h : \text{Lists(A)} \to \text{Lists(A} \times \text{A)}$ defined by $h(\langle x_1, ..., x_n)) = \langle(x_1, x_1), \ldots, (x_n, x_n)\rangle$.

$h(\langle x\rangle) = \langle(x,x)\rangle$

h(y :: L) = (y,y) :: h(L)

d. leaves: BinaryTrees(**N**) → **N** where leaves(*T*) is the number of leaves in the binary tree *T*.

$\text{leaves}((\langle\rangle) = 0$

$\text{leaves}(\text{tree}(\langle\rangle,a,\langle\rangle) = 1$

leave(tree(l, a, r)) = leaves(l) + leaves(r)