Section 1.2 Sets

A **set** is a collection of things.

If S is a set and x is a member or element of S we write $x \in S$. Otherwise we write $x \notin S$.

The set with elements $x_1, \ldots, x_n$ is denoted $\{x_1, \ldots, x_n\}$.

The empty set with no elements is denoted { } or ∅.

A set with one element is called a singleton. e.g., {a} is a singleton.

The set of *integers* is denoted by **Z**, the *natural numbers* {0, 1, …} by **N**, the *rational numbers* by **Q**, and the *real numbers* by **R**.

Equal Sets

Two sets A and B are equal, denoted A = B if they have the same elements.

e.g., {a, b, c} = {c, b, a} (no ordering)

e.g., {a, a, b, c} = {a, b, c} (no repetitions)

Sets can be described by properties that the elements satisfy. If *P* is a property, then the expression $\{x \mid P\}$ denotes the set of all x that satisfy *P*.

e.g., The set of odd natural numbers can be represented by the following equal sets.

$$\{x \mid x = 2k + 1 \text{ for some } k \in \mathbf{N}\} = \{1, 3, 5, \ldots\}.$$

Subsets

The set A is a **subset** of B, denoted $A \subset B$, means every element of A is an element of B.

e.g., $N \subset Z \subset Q \subset R$.

e.g., $S \subset S$ for any set S.

e.g., $\varnothing \subset S$ for any set S.

The **power set** of a set S, denoted power(S) is the set of all subsets of S.

e.g., power({a, b}) = {∅, {a}, {b}, {a, b}}.

Example(Comparing sets): Let A = {2k + 7 | k ∈ Z} and B = {4k + 3 | k ∈ Z}.

Quiz: Is A ⊂ B?

Answer: No. For example, 9 ∈ A but 9 ∉ B.

Quiz: Is B ⊂ A?

Answer: Yes. Let x ∈ B. Then x = 4k + 3 for some k ∈ Z. But we can write

$$x = 4k + 3 = 4k - 4 + 7 = 2(2k - 2) + 7$$

Since 2k – 2 ∈ Z, it follows that x ∈ A. Therefore B ⊂ A. QED.

Equality in terms of subsets: A = B iff A ⊂ B and B ⊂ A.

Example: Let A = {2k + 5 | k ∈ Z} and B = {2k + 3 | k ∈ Z}.

Show that A = B.

Proof: First show A ⊂ B. Let x ∈ A. Then x = 2k + 5 for some k ∈ Z. So we have

$$x = 2k + 5 = 2k + 2 + 3 = 2(k + 1) + 3$$

Since k + 1 ∈ Z, it follows that x ∈ B. Therefore A ⊂ B.

Now show the other direction B ⊂ A.

Proof: Class fill in proof as a quiz

Operations on Sets

Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Difference: $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.

Symmetric Difference: $A \oplus B = \{x \mid x \in A \text{ or } x \in B \text{ but not both}\}$

Note: $A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Universal Complement: Given a universe U and $A \subset U$, we write

$A' = U - A$.

Example: For each $n \in \mathbf{N}$ let $D_n = \{x \in \mathbf{N} \mid x \text{ divides } n\}$. So D_n is the set of positive divisors of n. Here are some expressions involving these sets.

$D_0 = \{1, 2, 3, \ldots\} = N - \{0\}$, $D_5 = \{1, 5\}$, $D_6 = \{1, 2, 3, 6\}$, and $D_9 = \{1, 3, 9\}$.

$D_5 \cup D_6 = \{1, 2, 3, 5, 6\}$

$D_5 \cap D_6 = \{1\}$

$D_9 - D_6 = \{9\}$

$D_5 \oplus D_6 = \{2, 3, 5, 6\}$

Let N be the universe. Then $D_0' = N - D_0 = \{0\}$, and $\{0\}' = D_0$.

Quiz (2 minutes): Draw a Venn diagram for three sets A, B, C with some areas shaded. Then find an expression to represent the shaded area.

Properties of Set Operations

Union and intersection are commutative, associative, and distribute over each other. There are many other properties too. For example,

Absorption: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$.

De Morgan's Laws: $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

Counting Sets

The *cardinality* of a set S is denoted by $|S|$. Two rules for counting finite sets are:

Inclusion-Exclusion or Union Rule: $| A \cup B | = | A | + | B | - | A \cap B |$.

Difference Rule: $| A - B | = | A | - | A \cap B |$.

Quiz (2 minutes): Find a rule for the union of three sets: $| A \cup B \cup C | = ?$

Quiz (3 minutes): Three programs use a collection of processors in the following way, where A, B, and C represent the sets of processors used by the three programs:

$$| A | = 20,\ | B | = 40,\ | C | = 60,\ | A \cap B | = 10,\ | A \cap C | = 8,\ | B \cap C | = 6.$$

If there are 100 processors available, what could $| A \cap B \cap C |$ be?

Answer: $100 \geq | A \cup B \cup C | = 20 + 40 + 60 - 10 - 8 - 6 + | A \cap B \cap C |$. So

$$| A \cap B \cap C | \leq 4.$$

Bags (Multisets) are like sets but can contain repeated elements. e.g., [t, o, o, t] = [o, t, t, o].

Union and intersection can be defined by taking the maximum and minimum occurrences of each element, respectively.

Quiz (2 minutes). Let A = [m, i, s, s, i, s, s, i, p, p, i] and B = [s, i, p, p, i, n, g].
What are $A \cup B$ and $A \cap B$?

Answer: $A \cup B$ = [m, i, s, s, i, s, s, i, p, p, i, n, g] and $A \cap B$ = [s, i, p, p, i].