Section 1.3 Ordered Structures

Tuples: Have order and can have repetitions. For example, (6, 7, 6) is a 3-tuple and ( ) is the empty tuple. We write $(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$ to mean $x_i = y_i$ for $1 \leq i \leq n$.

Cartesian Product: $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$. The definition extends to more than two sets. For example, $A \times B \times C = \{(x, y, z) \mid x \in A, y \in B, z \in C\}$.

Notation: $A^0 = \{(\ )\}$, $A^1 = \{(x) \mid x \in A\}$, and in general, $A^n = \{(x_1, \ldots, x_n) \mid x_i \in A\}$.

Quiz (1 minute). Does $(A \times B) \times C = A \times (B \times C)$?

Lists: Are like tuples but there is no random access. For example, <a, b, a, c> is a list with 4 elements and < > is the empty list.

- The operations on lists are *head*, *tail*, and *cons*. For example,

  head(<a, b, a, c>) = a, tail(<a, b, a, c>) = <b, a, c>, cons(a, <b, a, c>) = <a, b, a, c>.
- The set of lists whose elements are in A is denoted by lists(A).

Quiz (1 minute). For L = < <a>, b, <c, d> >, find head(L) and tail(L).

Strings: Are like lists, but are represented as juxtaposed elements from a given alphabet. For example, if A = {*a, b*}, then some strings over A are: *a, b, aa, ab, ba, bb, aaa, bbb*.

- The empty string is denoted by Λ.
- The concatenation of two strings is their juxtaposition. For example, the concatenation of *ab* and *bab* is *abbab*.
- For any string s we have $s\,\Lambda = \Lambda\, s = s$.
- If s is a string, s^n denotes the concatenation of s with itself n times. Also $s^0 = \Lambda$. For example, $(ab)^3 = ababab$.

Languages

A language is a set of strings, usually taken over some alphabet.

Notation: If A is an alphabet, then the set of all strings over A is denoted by A*.

Example. Some languages over A are: ∅, {L}, A, and A*.

Example. $\{ab^na \mid n \in \mathbf{N}\}$ = {*aa*, *aba*, *abba*, *abbba*, … } is a language over {a, b}.

Language Operations

Let L and M be languages. The product of L and M, denoted LM, is the language

$$LM = \{st \mid s \in L \text{ and } t \in M\}.$$

- Notation: $L^0 = \{\Lambda\}$; and $L^n = \{s_1 \ldots s_n \mid s_i \in L\}$.

Quiz (1 minute): What are the products L ∅ and L{Λ}?

Quiz (1 minute): Solve for L in the equation {Λ, *a, b*}L = {Λ, *a, b, aa, ba, aba, bba*}.

The closure L* is the set of all possible concatenations of strings in L. So

$$L^* = L^0 \cup L^1 \cup \ldots \cup L^n \cup \ldots$$

Quiz (1 minute): What are $\{\Lambda\}^*$ and $\varnothing^*$?

Example. Examine the structure of an arbitrary string $x \in L^*(ML)^*$.

A solution: Use the definitions to write x in terms of strings in L and M.

1. Since $x \in L^*(ML)^*$, it follows that $x = uv$ where $u \in L^*$ and $v \in (ML)^*$.
2. Since $u \in L^*$, either $u = \Lambda$ or $u = s_1 \ldots s_n$ for some n where $s_i \in L$.
3. Since $v \in (ML)^*$, either $v = \Lambda$ or $v = r_1t_1 \ldots r_kt_k$ for some k where $r_i \in M$ and $t_i \in L$.

So x has one of four forms: Λ, $s_1 \ldots s_n$, $r_1t_1 \ldots r_kt_k$, or $s_1 \ldots s_n r_1t_1 \ldots r_kt_k$.

Relations

A *relation* is a set of tuples. If *R* is a relation and $(x_1, \ldots, x_n) \in R$, we write $R(x_1, \ldots, x_n)$.

We can usually represent a relation as a subset of some cartesian product.

Example. Let $R = \{(0, 0), (1, 1), (4, 2), (9, 3), \ldots, (n^2, n), \ldots\} = \{(n^2, n) \mid n \in N\}$. We might call *R* the "is square of" relation on **N**. Notice also that $R \subset \mathbf{N} \times \mathbf{N}$.

Notation: If *R* is binary, we can use infix to represent pairs in R. For example, from the previous example, we have $(9, 3) \in R$. So we can also write

R(9, 3)   or   9 R 3   or   9 is square of 3.

Relational Databases

A *relational database* is a relation where the indexes of a tuple have associated names called *attributes*.

Example. Let Students = {(*x, y, z*) | *x* is a Name, *y* is a Major, and *z* is Credits}.

- Who are the people majoring in CS?

  $\{x \mid (x, \text{CS}, z) \in \text{Students}, \text{ for some } z\}$.

  Note: We need "for some *z*" to indicate that *z* is a variable.
- How many math majors are upper division students?

  $|\ \{x \mid (x, \text{math}, z) \in \text{Students and } z \geq 90\}\ |$.
- What is the major of AbeLincoln?

  $\{y \mid (\text{AbeLincoln}, y, z) \in \text{Students}, \text{ for some } z\}$.
- What is the history department database of names and their credits?

  $\{(x, z) \mid (x, \text{history}, z) \in \text{Students}\}$.

Counting Tuples (or *strings* or *lists*)

Product Rule: $|A \times B| = |A||B|$ and $|A^n| = |A|^n$.

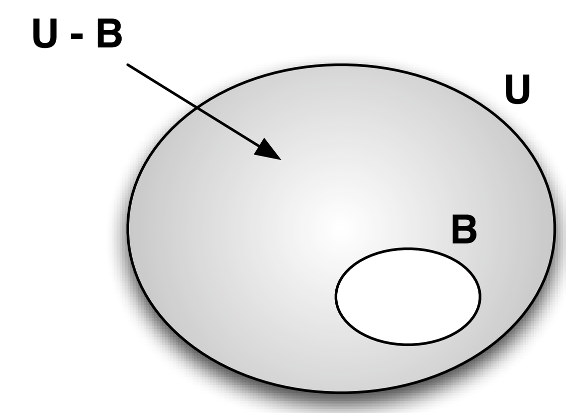
Example. If A = {*a, b*} and B = {1, 2, 3}, then

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}.$$

So $|A \times B| = 6 = (2)(3) = |A|\,|B|$.

Example: Count the number of strings of length 8 over A = {a, b, c} that begin with either a or c and have at least one b.

A Solution: Split the problem up into easier problems and combine the results (divide and conquer). Let U be the universe consisting of the strings over A of length 8 that begin with either a or c. Let B be the subset of U consisting of strings with no b's. Then the set of strings to count is U – B, as pictured.



It is easy to calculate the cardinality of U – B:

$$|U - B| = |U| - |U \cap B| = |U| - |B| \text{ (since B is a subset of U)}$$

It is also easy to count *U* because it has the same cardinality as $\{a, c\} \times A^7$, which is

$$|\{a, c\} \times A^7| = |\{a, c\}|\,|A^7| = |\{a, c\}|\,|A|^7 = (2)3^7.$$

It is also easy to count B because it has the same cardinality as the set $\{a, c\}^8$, which is

$$|\{a, c\}^8| = |\{a, c\}|^8 = 28.$$

So we have the answer:

$$|U - B| = |U| - |U \cap B| = |U| - |B| = (2)3^7 - 28, \text{ which is } 4118.$$