Section 2.1 Functions: Definitions and Examples

A function f from A to B associates each element of A with exactly one element of B. Write $f : A \Rightarrow B$ and call A the *domain* and B the *codomain*.

Write $f(x) = y$ to mean f associates $x \in A$ with $y \in B$. Say, "f of x is y" or "f maps x to y."

- If $C \subset A$, the *image* of C is the set $f(C) = \{f(x) \mid x \in C\}$.
- The *range* of f is the image of A. We write $\text{range}(f) = f(A) = \{f(x) \mid x \in A\}$.
- If $D \subset B$, the pre-image (or inverse image) of D is the set $f^{-1}(D) = \{x \mid f(x) \in D\}$.

Example: The picture shows a function $f : A \Rightarrow B$ with domain $A = \{a, b, c, d\}$ and codomain $B = \{1, 2, 3\}$.

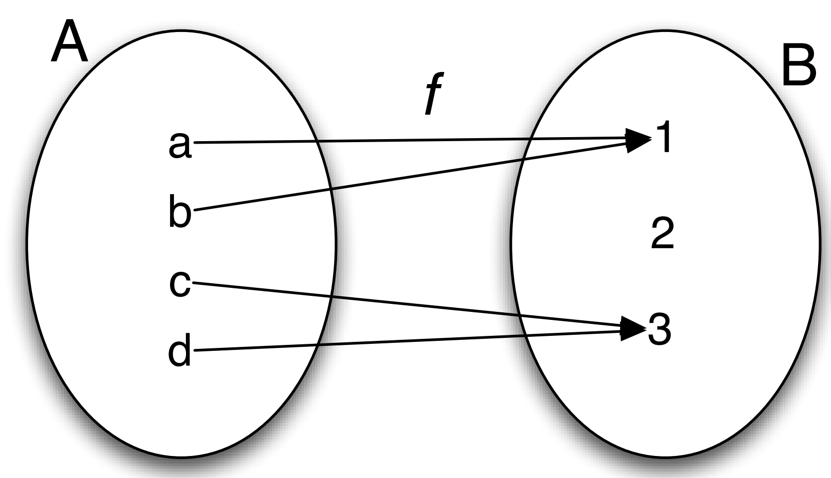
We have: $f(a) = f(b) = 1$ and $f(c) = f(d) = 3$.

Some sample sets are: $\text{range}(f) = \{1, 3\}$,

$$f(\{a, b\}) = \{1\},$$

$$f^{-1}(\{2\}) = \varnothing,$$

$$f^{-1}(\{1, 2, 3\}) = \{a, b, c, d\}.$$



Example: Let $f : \mathbf{Z} \Rightarrow \mathbf{Z}$ be defined by $f(x) = 2x$. Let E and O be the sets of even and odd integers, respectively. Some sample sets are:

$$\text{range}(f) = f(Z) = E;\ f(E) = \{4k \mid k \in \mathbf{Z}\};$$

$$f(O) = \{4k + 2 \mid k \in Z\};\ f^{-1}(E) = \mathbf{Z};\ f^{-1}(O) = \varnothing.$$

Quiz (2 minutes): Let $f : \mathbf{N} \Rightarrow \mathbf{N}$ by $f(x) =$ if x is odd then $x + 1$ else x. Find each set, where E and O are the even and odd natural numbers. $\text{range}(f)$, $f(E)$, $f(O)$, $f^{-1}(N)$, $f^{-1}(E)$, $f^{-1}(O)$. ***Answers***: E, E, $E - \{0\}$, **N**, **N**, $\varnothing$.

Floor and Ceiling

The floor and ceiling functions have type **R** ⇒ **Z**, where floor(x) is the closest integer less than or equal to x and ceiling(x) is the closest integer greater than or equal to x. e.g., floor(2.6) = 2, floor(–2.1) = –3, ceiling(2.6) = 3, and ceiling(–2.1) = –2.

Notation: $\lfloor x \rfloor$ = floor(x) and $\lceil x \rceil$ = ceiling(x).

There are many interesting properties, all of which are easy to prove.

Example: $-\lfloor x \rfloor = \lceil -x \rceil$.

Proof: If $x \in \mathbf{Z}$, then $-\lfloor x \rfloor = -x = \lceil -x \rceil$. If $x \in \mathbf{Z}$, there is an $n \in \mathbf{Z}$ such that $n < x < n + 1$, which gives $\lfloor x \rfloor = n$. Multiply the inequality by –1 to get $-n > -x > -(n + 1)$, which gives $\lceil -x \rceil = -n$. So $-\lfloor x \rfloor = -n = \lceil -x \rceil$. QED.

Quiz (1 minute): Show that $\lceil x + 1 \rceil = \lceil x \rceil + 1$.

Proof: There is an integer n such that $n < x \leq n + 1$. Add 1 to get $n + 1 < x + 1 \leq n + 2$. So $\lceil x + 1 \rceil = n + 2$ and $\lceil x \rceil = n + 1$. Therefore, $\lceil x + 1 \rceil = \lceil x \rceil + 1$. QED.

Greatest Common Divisor (gcd)

If x and y are integers, not both zero, then gcd(x, y) is the largest integer that divides x and y.

For example, gcd(12, 15) = 3, gcd(–12, –8) = 4.

Properties of gcd

- gcd(a, b) = gcd(b, a) = gcd(a, –b).
- gcd(a, b) = gcd(b, a – bq) for any integer q.
- gcd(a, b) = ma + nb for some m, $n \in \mathbf{Z}$.
- If d | ab and gcd(d, a) = 1, then d | b.

Division algorithm

For a, b ∈ Z, b ≠ 0 there are unique q, r ∈ **Z** such that $a = bq + r$ where $0 \le r < |b|$.

Euclid's algorithm for finding gcd(a, b)

Assume a, b ∈ **N**, not both zero.

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while b > 0 do
      find q, r so that a = bq + r and 0 ≤ r < b;
      a := b;
      b := r
od
Output(a)
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Example: Find gcd(189, 33)

$189 = 33 \cdot 5 + 24$

$33 = 24 \cdot 1 + 9$

$9 = 6 \cdot 1 + 3$

$6 = 3 \cdot 2 + 0$

Output(3)

Quiz: How many loop iterations in Euclid's algorithm to find gcd(117, 48)?

The mod Function

For a, b ∈ Z with b > 0 apply the division algorithm to get $a = bq + r$ with $0 \le r < b$. *The remainder r is the value of the mod function applied to a and b*. To get a formula for r in terms of a and b solve the equation for $q = a/b - r/b$. Since q ∈ **Z** and $0 \le r/b < 1$, it follows that $q = \lfloor a/b \rfloor$. So we have $r = a - bq = a - b \cdot \lfloor a/b \rfloor$.

The value of r is denoted "*a* mod *b*". So if a, b ∈ **Z** with b > 0 then

$$a \bmod b = a - b \cdot \lfloor a/b \rfloor .$$

Quiz (1 minute): It is 2am in Paris. What time is it in Portland (9 hours difference)?

Answer: (12 hr clock):$(2 - 9) \bmod 12 = (-7) \bmod 12 = -7 - 12 \lfloor -7/12 \rfloor = -7 - 12(-1) = 5$.

(24 hr clock) $(2 - 9) \bmod 24 = (-7) \bmod 24 = -7 - 24 \lfloor -7/24 \rfloor = -7 - 24(-1) = 17$.

Quiz (1 minute): What are the elements in the set $\{x \bmod 5 \mid x \in Z\}$?

Answer: $\{0, 1, 2, 3, 4\}$.

Notation: $N_n = \{0, 1, \ldots, n - 1\}$. If *n* is fixed, the range of values for x mod n is $\mathbf{N_n}$.

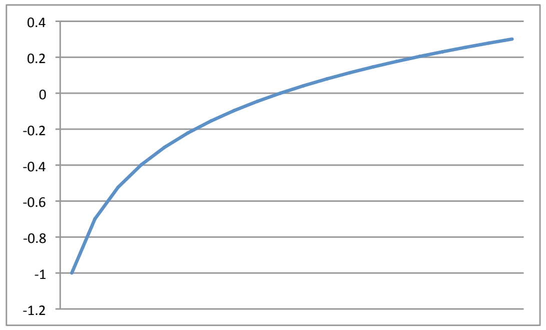
Quiz (1 minute): Convert 13 to binary.

Answer: For any $x \geq 0$ we can write $x = 2 \lfloor x/2 \rfloor + x \bmod 2$. So $13 = 2 \cdot 6 + 1$

$$6 = 2 \cdot 3 + 0$$
$$3 = 2 \cdot 1 + 1$$
$$1 = 2 \cdot 0 + 1$$

The remainders from bottom to top are the binary digits from left to right: 1101.

Log Function



If x and b are positive real numbers and $b > 1$, then $\log_b x = y$ means $b^y = x$. The graph of the log function is pictured at the right.

Notice that log is an increasing function: $s < t$ implies $\log_b s < \log_b t$.

Properties of log: $\log_b 1 = 0$, $\log_b b = 1$, $\log_b b^x = x$,

$\log_b xy = \log_b x + \log_b y$, $\log_b x^y = y\log_b x$, and $\log_a x = (\log_a b)(\log_b x)$.

Quiz (3 minutes): Estimate $\log_2(5^2 2^5)$ by finding upper and lower bounds.

One answer: $\log_2(5^2 2^5) = \log_2(5^2) + \log_2(2^5) = 2\log_2 5 + 5\log_2 2 = 2\log_2 5 + 5$.

Since $4 < 5 < 8$, we can apply $\log_2$ to the inequality to get $2 < \log_2 5 < 3$.

Multiply by 2 to get $4 < 2\log_2 5 < 6$. So $9 < \log_2(5^2 2^5) < 11$.

Another answer: Use $16 < 5^2 < 32$. Then $4 < \log_2(5^2) < 5$. So $9 < \log_2(5^2 2^5) < 10$.