

Section 2.2 Constructing Functions

Composition

If $g : A \rightarrow B$ and $f : B \rightarrow C$, then the expression $f(g(x))$ makes sense and the composition of f and g , denoted $f \circ g : A \rightarrow C$, is defined by $(f \circ g)(x) = f(g(x))$.

Some examples:

$$\text{floor}(\log_2 20) = \text{floor}(4.3219) = 4$$
$$\text{ceiling}(\log_2 20) = \text{ceiling}(4.3219) = 5$$
$$\text{head}(\text{tail}(\langle a, b, c \rangle)) = \text{head}(\langle b, c \rangle) = b.$$

The Sequence and Distribute Functions

$\text{seq} : \mathbf{N} \rightarrow \text{lists}(\mathbf{N})$ by $\text{seq}(n) = \langle 0, 1, \dots, n \rangle$.

$\text{dist} : A \times \text{lists}(B) \rightarrow \text{lists}(A \times B)$ by $\text{dist}(a, \langle x_1, \dots, x_k \rangle) = \langle (a, x_1), \dots, (a, x_k) \rangle$.

Example. How can we compute $f : \mathbf{N} \rightarrow \text{lists}(\mathbf{N})$ defined by $f(x) = \langle 1, 2, \dots, x + 1 \rangle$?

Solution: Let $f(x) = \text{tail}(\text{seq}(x + 1))$.

Quiz (3 minutes): How can we compute $f : \mathbf{N} \rightarrow \text{lists}(\mathbf{N} \times \mathbf{N})$ defined by

$$f(x) = \langle (x, 1), (x, 2), \dots, (x, x) \rangle ?$$

Solution: Let $f(x) = \text{dist}(x, \text{tail}(\text{seq}(x)))$, or equivalently, $\text{tail}(\text{dist}(x, \text{seq}(x)))$.

Map Function

The map function takes a function f and a list and returns the list of values obtained by applying f to each element in the list. So map is defined by

$$\text{map}(f, \langle x_1, \dots, x_k \rangle) = \langle f(x_1), \dots, f(x_k) \rangle.$$

Quiz (2 minutes): Evaluate the expression $\text{map}(\text{floor} \circ \log_2, \text{tail}(\text{seq}(8)))$.

Answer: $\langle 0, 1, 1, 2, 2, 2, 2, 3 \rangle$

Example: Find a definition for $f : \mathbf{N} \rightarrow \text{lists}(\mathbf{N})$ defined by $f(n) = \langle 0, 2, 4, \dots, 2n \rangle$.

Solution: Transform the list into a composition of known functions.

$$\begin{aligned} f(n) &= \langle 0, 2, 4, \dots, 2n \rangle \\ &= \langle 2 \cdot 0, 2 \cdot 1, 2 \cdot 2, \dots, 2 \cdot n \rangle \\ &= \text{map}(\cdot, \langle (2, 0), (2, 1), \dots, (2, n) \rangle) \\ &= \text{map}(\cdot, \text{dist}(2, \langle 0, 1, 2, \dots, n \rangle)) \\ &= \text{map}(\cdot, \text{dist}(2, \text{seq}(n))). \end{aligned}$$

So define $f(n) = \text{map}(\cdot, \text{dist}(2, \text{seq}(n)))$.

Example: Find a definition for $f : \mathbf{N} \rightarrow \text{lists}(\mathbf{N})$ defined by $f(n) = \langle 5, 9, 13, \dots, 5 + 4n \rangle$

$$\begin{aligned} \text{Solution: } f(n) &= \langle 5, 9, 13, \dots, 5 + 4n \rangle \\ &= \langle 5 + 4 \cdot 0, 5 + 4 \cdot 1, 5 + 4 \cdot 2, \dots, 5 + 4 \cdot n \rangle \\ &= \text{map}(+, \langle (5, 4 \cdot 0), (5, 4 \cdot 1), (5, 4 \cdot 2), \dots, (5, 4 \cdot n) \rangle) \\ &= \text{map}(+, \text{dist}(5, \langle 4 \cdot 0, 4 \cdot 1, 4 \cdot 2, \dots, 4 \cdot n \rangle)) \\ &= \text{map}(+, \text{dist}(5, \text{map}(\cdot, \langle (4, 0), (4, 1), (4, 2), \dots, (4, n) \rangle))) \\ &= \text{map}(+, \text{dist}(5, \text{map}(\cdot, \text{dist}(4, \langle 0, 1, 2, \dots, n \rangle)))) \\ &= \text{map}(+, \text{dist}(5, \text{map}(\cdot, \text{dist}(4, \text{seq}(n))))) \end{aligned}$$

So we can define $f(n) = \text{map}(+, \text{dist}(5, \text{map}(\cdot, \text{dist}(4, \text{seq}(n)))))$.

Quiz (3 minutes): Find a definition for $f : \mathbf{N}^3 \rightarrow \text{lists}(\mathbf{N})$ defined by $f(a, b, n) = \langle a, a + b, a + 2b, \dots, a + nb \rangle$.

Solution: This is just an abstraction of the previous example, where $a = 5$ and $b = 4$. So we can define $f(a, b, n) = \text{map}(+, \text{dist}(a, \text{map}(\cdot, \text{dist}(b, \text{seq}(n)))))$.