Section 3.1 Inductively Defined Sets

To define a set S *inductively* is to do three things:

Basis: Specify one or more elements of S.

Induction: One or more rules to construct elements of S from existing elements of S.

Closure: Specify that no other elements are in S (always assumed).

Note: The basis elements and the induction rules are called constructors.

Example 1: Find an inductive definition for S = {3, 16, 29, 42, …}.

Solution: *Basis*: $3 \in S$.

Induction: If $x \in S$ then $x + 13 \in S$.

The constructors are 3 and the operation of adding 13. Also, without closure, many sets would satisfy the basis and induction rule. e.g., $3 \in \mathbf{Z}$ and $x \in \mathbf{Z}$ implies $x + 13 \in \mathbf{Z}$.

Example 2: Find an inductive definition for S = {3, 4, 5, 8, 9, 12, 16, 17, 20, 24, 33,…}.

Solution: To simplify things we might try to "divide and conquer" by writing S as the union of more familiar sets as follows:

$$S = \{3, 5, 9, 17, 33, \dots\} \cup \{4, 8, 12, 16, 20, 24, \dots\}.$$

Basis: $3, 4 \in S$.

Induction: If $x \in S$ then (if x is odd then $2x - 1 \in S$ else $x + 4 \in S$).

Example 3: Describe the set S defined inductively as follows:

Basis: $2 \in S$;

Induction: $x \in S$ implies $x \pm 3 \in S$.

Solution: $S = \{2, 5, 8, 11, \dots\} \cup \{-1, -4, -7, -10, \dots\}$.

Example 4: Find an inductive definition for S = {Λ, ac, aacc, aaaccc, …} = $\{a^n c^n \mid n \in \mathbf{N}\}$.

Solution: *Basis*: Λ ∈ S.

Induction: If x ∈ S then axc ∈ S.

Example 5: Find an inductive definition for S = $\{a^{n+1} b c^n \mid n \in N\}$.

Solution: *Basis*: ab ∈ S.

Induction: If x ∈ S then axc ∈ S.

Example 6: Describe the set S defined by: Basis: a, b ∈ S

Induction: x ∈ S implies ƒ(x) ∈ S.

Solution: S = {a, ƒ(a), ƒ(ƒ(a)), …} ∪ {b, ƒ(b), ƒ(ƒ(b)), …}, which could also be written as

$$S = \{f^n(a) \mid n \in N\} \cup \{f^n(b) \mid n \in N\} = \{f^n(x) \mid x \in \{a, b\} \text{ and } n \in N\}.$$

Example 7: Describe the set S defined by: Basis: 〈 0 〉 ∈ S

Induction: x ∈ S implies cons(1, x) ∈ S.

Solution: S = { 〈 0 〉 , 〈 1, 0 〉 , 〈1, 1, 0 〉 , …}.

Infix notation

cons(h, t) = h :: t. Associate to the right. e.g., x :: y :: z = x :: (y :: z).

Example 8: Find an inductive definition for S = { 〈 〉 , 〈 a, b 〉 , 〈a, b, a, b 〉 , …}.

Solution: *Basis*: 〈 〉 ∈ S.

Induction: x ∈ S implies a :: b :: x ∈ S.

Example 9: Find an inductive definition for S = { 〈 〉 , 〈 〈 〉 〉 , 〈 〈 〈 〉 〉 〉 , …}.

Solution: *Basis*: 〈 〉 ∈ S.

Induction: x ∈ S implies x :: 〈 〉 ∈ S.

Notation for Binary Trees

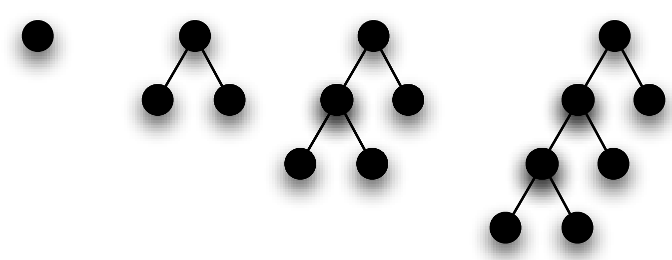
Let t(L, x, R) denote the tree with root x, left subtree L, and right subtree R. Let ⟨ ⟩ denote the empty binary tree. If T = t(L, x, R), then root(T) = x, left(T) = L, and right(T) = R.

Example 10. Describe the set S defined inductively as follows:

Basis: t(⟨ ⟩ , •, ⟨ ⟩) $\in$ S.

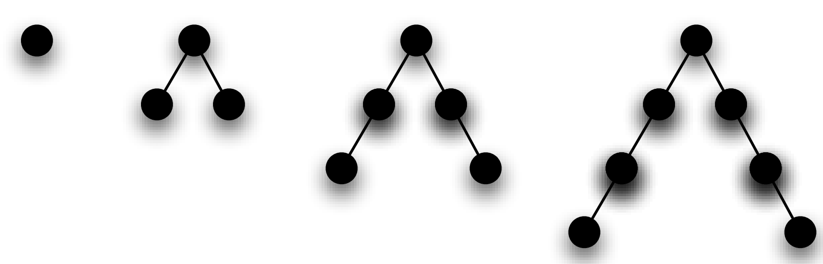
Induction: T $\in$ S implies t(T, •, t(⟨ ⟩ , •, ⟨ ⟩)) $\in$ S.

Solution (picture): The first few trees constructed from the definition are pictured as follows:



and so on...

Example 11. Find an inductive definition for the set S of binary trees indicated by the following picture.



and so on...

Solution: *Basis*: t(⟨ ⟩ , •, ⟨ ⟩) $\in$ S.

Induction: T $\in$ S implies t(t(left(T), •, ⟨ ⟩), •, t(⟨ ⟩ , •, right(T))) $\in$ S.

Example 12: Find an inductive definition for the set $S = \{a\}^* \times \mathbf{N}$.

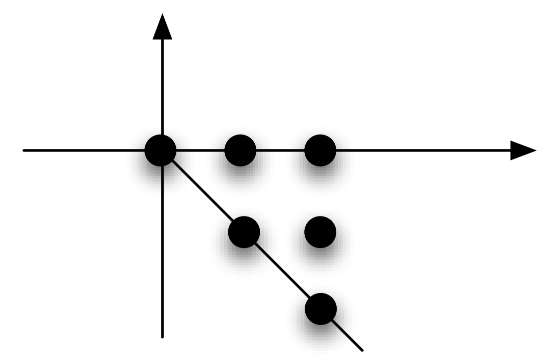
Solution: Basis: $(\Lambda, 0) \in S$.

Induction: $(s, n) \in S$ implies $(as, n), (s, n + 1) \in S$.

Example 13: Find an inductive definition for the set $S = \{(x, -y) \mid x, y \in \mathbf{N} \text{ and } x \geq y\}$.

Solution: To get an idea about S we can write out a few tuples:

(0, 0), (1, 0), (1, –1), (2, 0), (2, –1), (2, –2), and so on.



We can also get an idea about S by graphing a few points, as indicated in the picture.

One solution can be written as follows:

Basis: $(0, 0) \in S$.

Induction: $(x, y) \in S$ implies $(x + 1, y), (x + 1, y - 1) \in S$.

Notice that this definition constructs some repeated points.

For example, (2, –1) is constructed twice.

Quiz (2 minutes): Try to find a solution that does not construct repeated elements.

Solution: We might use two separate rules. One rule to construct the diagonal points and one rule to construct horizontal lines that start at the diagonal points.

Basis: $(0, 0) \in S$.

Induction: 1. $(x, y) \in S$ implies $(x + 1, y) \in S$.

2. $(x, -x) \in S$ implies $(x + 1, -(x + 1)) \in S$.