Section 4.1 Properties of Binary Relations

A *binary relation* R over a set A is a subset of $A \times A$. If $(x, y) \in R$ we also write $x\ R\ y$.

Example. Some sample binary relations over $A = \{0, 1\}$ are

$$\varnothing, \quad A \times A, \quad \text{eq} = \{(0, 0), (1, 1)\}, \quad \text{less} = \{(0, 1)\}.$$

Definitions: Let R be a binary relation over a set A.

- R is *reflexive* means: $x\ R\ x$ for all $x \in A$.
- R is *symmetric* means: $x\ R\ y$ implies $y\ R\ x$ for all $x, y \in A$.
- R is *transitive* means: $x\ R\ y$ and $y\ R\ z$ implies $x\ R\ z$ for all $x, y, z \in A$.
- R is *irreflexive* means: $(x, x) \notin R$ for all $x \in A$.
- R is *antisymmetric* means: $x\ R\ y$ and $y\ R\ x$ implies $x = y$ for all $x, y \in A$.

Example/Quiz. Describe the properties that hold for the four sample relations:

1. $\varnothing$.
2. $A \times A$.
3. eq = {(0, 0), (1, 1)}.
4. less = {(0, 1)}.

Answer:

1. symmetric, transitive, irreflexive, antisymmetric.
2. reflexive, symmetric, transitive.
3. reflexive, symmetric, transitive, antisymmetric.
4. irreflexive, transitive, antisymmetric.

Composition: If R and S are binary relations, then the *composition* of R and S is

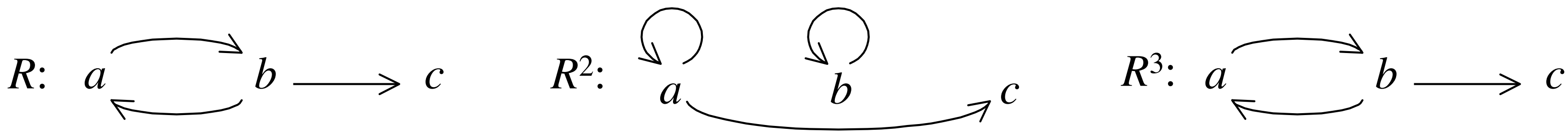
$$R \circ S = \{(x, z) \mid x\ R\ y \text{ and } y\ S\ z \text{ for some } y\}.$$

Example. eq $\circ$ less = less.

Quiz.
1. $R \circ \varnothing = ?$
2. isMotherOf $\circ$ isFatherOf = ?
3. isSonOf $\circ$ isSiblingOf = ?

Answer.
1. $\varnothing$.
2. IsPaternalGrandmotherOf.
3. isNephewOf.

Example (digraph representations). Let $R = \{(a, b), (b, a), (b, c)\}$ over $A = \{a, b, c\}$. Then R, $R^2 = R \circ R$, and $R^3 = R^2 \circ R$ can be represented by the following directed graphs:



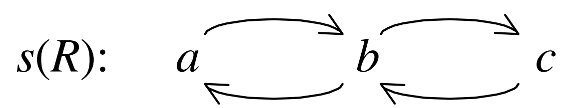
Closures. The closure of R with respect to a property is the smallest binary relation containing R that satisfies the property. We have the following three notations and results.

- The *reflexive closure* of R is $r(R) = R \cup \text{Eq}$, where Eq is the equality relation on A.
- The *symmetric closure* of R is $s(R) = R \cup R^c$, where $R^c = \{(b, a) \mid a\ R\ b\}$.
- The *transitive closure* of R is $t(R) = R \cup R^2 \cup R^3 \cup \ldots$.
  Note: If $|A| = n$, then $t(R) = R \cup R^2 \cup \ldots \cup R^n$.

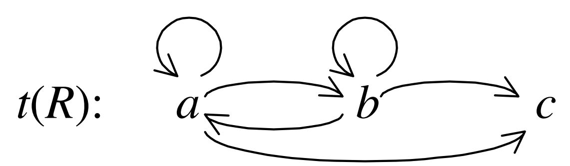
Example. $R = \{(a, b), (b, a), (b, c)\}$ over $A = \{a, b, c\}$. Calculate the three closures of R.

$r(R) = R \cup \text{Eq}$
$= \{(a, b), (b, a), (b, c), (a, a), (b, b), (c, c)\}$.

$s(R) = R \cup R^c = \{(a, b), (b, a), (b, c), (c, b)\}$.



$t(R) = R \cup R^2 \cup R^3$
$= \{(a, b), (b, a), (b, c), (a, a), (b, b), (a, c)\}$.



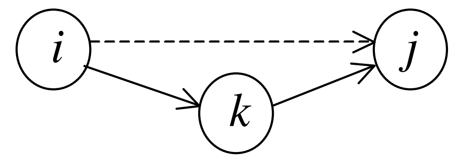
Quiz (3 minutes). Let $R = \{(x, x + 1) \mid x \in \mathbf{Z}\}$. Find $t(R)$, $rt(R)$, and $st(R)$.

Solution: $t(R)$ is $<$
$rt(R)$ is $\leq$
$st(R)$ is $\neq$.

Path Problem (*Is there a path from i to j*?)
Let $R = \{(1, 2), (2, 3), (3, 4)\}$. We can represent R as an adjacency matrix M where M_{ij} is 1 if $i\ R\ j$ and 0 otherwise. If we want an answer to our question, it would be nice to have the adjacency matrix for $t(R)$. Then we could simply check whether $M_{ij} = 1$.

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Warshall's algorithm computes the matrix for $t(R)$. It constructs edge (i, j) if it finds edges (i, k) and (k, j), as pictured.



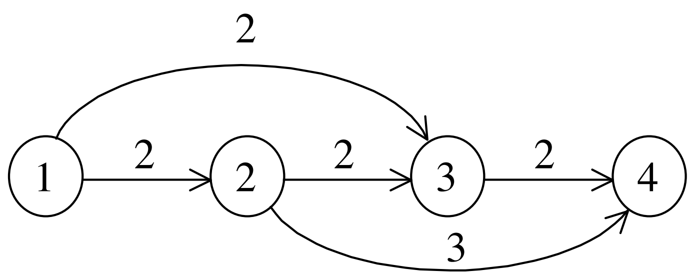
for k := 1 to n **for** i := 1 to n **for** j := 1 to n **do**
  if $M_{ik} = M_{kj} = 1$ **then** $M_{ij} := 1$.

Example. The following trace shows M after each k-loop, where the rightmost M contains the adjacency matrix of $t(R)$.

$k = 1$ (no change)		$k = 2$ $M_{13} := (M_{12} = M_{23} = 1)$		$k = 3$ $M_{14} := (M_{13} = M_{34} = 1)$ $M_{24} := (M_{23} = M_{34} = 1)$		$k = 4$ (no change)
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\to$	$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\to$	$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$\to$	$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Path Problem (*What is the length of the shortest path from i to j*?)
Suppose we have nonnegative weights assigned to each edge. Modify the adjacency matrix M so that M_{ij} is the weight on edge (i, j), $M_{ii} = 0$, and all other entries are .

Example.



$$M = \begin{pmatrix} 0 & 2 & 2 & \\ & 0 & 2 & 3 \\ & & 0 & 2 \\ & & & 0 \end{pmatrix}$$

Floyd's algorithm computes $t(R)$ *and* the lengths of the shortest paths.

for $k := 1$ to n **for** $i := 1$ to n **for** $j := 1$ to n **do**
$M_{ij} := \min\{M_{ij}, M_{ik} + M_{kj}\}$.

Example. The following trace shows M after each k-loop, where the rightmost M contains $t(R)$ and shortest path lengths.

$k = 1$	$k = 2$	$k = 3$	$k = 4$
no change	$M_{14} := \min\{M_{14}, M_{12}+M_{24}\}$	$M_{14} := \min\{M_{14}, M_{13}+M_{34}\}$	no change

$$\begin{pmatrix} 0 & 2 & 2 & \\ & 0 & 2 & 3 \\ & & 0 & 2 \\ & & & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 2 & 5 \\ & 0 & 2 & 3 \\ & & 0 & 2 \\ & & & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 2 & 4 \\ & 0 & 2 & 3 \\ & & 0 & 2 \\ & & & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 2 & 4 \\ & 0 & 2 & 3 \\ & & 0 & 2 \\ & & & 0 \end{pmatrix}$$

Path Problem (*What is a shortest path from i to j*?)

Modify Floyd by adding a path matrix P, where $P_{ij} = 0$ means edge (i, j) is the shortest path from i to j (if $M_{ij} \neq$), and $P_{ij} = k$ means a shortest path from i to j passes through k.

Floyd's modified algorithm computes $t(R)$, shortest path length, *and* the shortest path information P. (P is initalized to all zeros.)

for k := 1 to n **for** i := 1 to n **for** j := 1 to n *do*
  if $M_{ik} + M_{kj} < M_{ij}$ **then**
    $M_{ij} := M_{ik} + M_{kj}$;
    $P_{ij} := k$
  fi

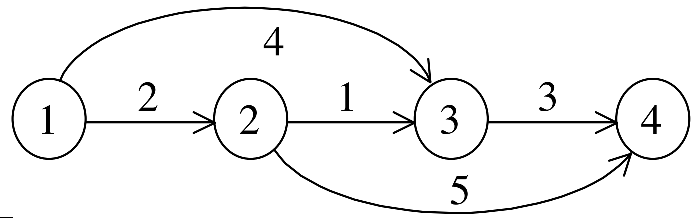
Example. For the previous example, the following trace shows M and P after each k-loop.

	$k = 1$ no change	$k = 2$ $M_{14} := M_{12}+M_{24}$ $P_{14} := 2$	$k = 3$ $M_{14} := M_{13}+M_{34}$ $P_{14} := 3$	$k = 4$ no change

$$M = \begin{bmatrix}0&2&2&\\ &0&2&3\\ & &0&2\\ & & &0\end{bmatrix} \quad \begin{bmatrix}0&2&2&5\\ &0&2&3\\ & &0&2\\ & & &0\end{bmatrix} \quad \begin{bmatrix}0&2&2&4\\ &0&2&3\\ & &0&2\\ & & &0\end{bmatrix} \quad \begin{bmatrix}0&2&2&4\\ &0&2&3\\ & &0&2\\ & & &0\end{bmatrix}$$

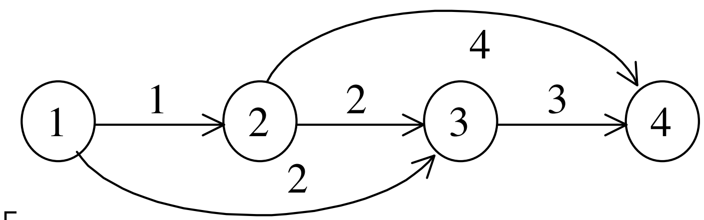
$$P = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&0&0&2\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&0&0&3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \quad \begin{bmatrix}0&0&0&3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Quiz (2 minutes). Use your wits to find a P matrix for the pictured graph.



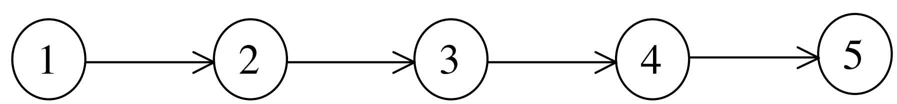
Answer. Either $\begin{bmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Quiz (2 minutes). Use your wits to find a P matrix for the pictured graph.



Answer. Either $\begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Quiz (3 minutes). How many possible P matrices for the pictured graph?



Answer. $\begin{bmatrix} 0 & 0 & 2 & x & y \\ 0 & 0 & 0 & 3 & z \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Where $x \in \{2, 3\}$, $y \in \{2, 3, 4\}$ and $z \in \{3, 4\}$. So there are 12 possible P matrices.