Section 4.2 Equivalence Relations

A binary relation is an *equivalence relation* if it has the three properties
reflexive, symmetric, and transitive (RST).

Examples. **a**. Equality on any set.
b. $x \sim y$ iff $|x| = |y|$ over the set of strings $\{a, b, c\}^*$.
c. $x \sim y$ iff x and y have the same birthday over the set of people.

Example. For any set of arithmetic expressions E let $e_1 \sim e_2$ iff e_1 and e_2 have the same value for any assignment to the variables. Then $\sim$ is RST. e.g., $4x + 2 \sim 2(2x + 1)$.

Quiz (2 minutes). Which of the relations are RST?
a. $x \; R \; y$ iff $x \leq y$ or $x > y$ over $\mathbf{Z}$.
b. $x \; R \; y$ iff $|x - y| \leq 2$ over $\mathbf{Z}$.
c. $x \; R \; y$ iff x and y are both even over $\mathbf{Z}$.

Answers. Yes, No, No.

Intersection Property. If E and F are RST over A, then $E \cap F$ is RST over A.

Example. Let $x \sim y$ iff x and y have the same birthday and the same family name. Then $\sim$ is RST since it is the intersection of two RSTs.

RSTs from functions (Kernel Relations). Any function $f : A \rightarrow B$ defines an RST on the set A by letting $x \sim y$ iff $f(x) = f(y)$.

Example. Let $x \sim y$ iff $x \bmod n = y \bmod n$ over any set S of integers. Then $\sim$ is an RST because it is the kernel relation of the function $f : S \rightarrow \mathbf{N}$ defined by $f(x) = x \bmod n$.

Example. Let $x \sim y$ iff $x + y$ is even over $\mathbf{Z}$. Then $\sim$ is RST because $x + y$ is even iff x and y are both even or both odd iff $x \bmod 2 = y \bmod 2$. So $\sim$ is the kernel relation of the Function $f : \mathbf{Z} \rightarrow \mathbf{N}$ defined by $f(x) = x \bmod 2$.

Equivalence Classes

If R is RST over A, then for each $a \in A$ the *equivalence class* of a, denoted $[a]$, is the set

$$[a] = \{x \mid x\ R\ a\}.$$

Property: For every pair $a, b \in A$ we have either $[a] = [b]$ or $[a] \cap [b] = \varnothing$.

Example. Suppose $x \sim y$ iff $x \bmod 3 = y \bmod 3$ over **N**. Then the equivalence classes are,

$[0] = \{0, 3, 6, \ldots\} = \{3k \mid k \in \mathbf{N}\}$
$[1] = \{1, 4, 7, \ldots\} = \{3k + 1 \mid k \in \mathbf{N}\}$
$[2] = \{2, 5, 8, \ldots\} = \{3k + 2 \mid k \in \mathbf{N}\}$.

Notice also, for example, that $[0] = [3] = [6]$ and $[1] \cap [2] = \varnothing$.

A Partition of a set is a collection of nonempty disjoint subsets whose union is the set.

Example. From the previous example, the sets [0], [1], [2] form a partition of **N**.

Theorem (*RSTs and Partitions*). Let A be a set. Then the following statements are true.

1. The equivalence classes of any RST over A form a partition of A.
2. Any partition of A yields an RST over A, where the sets of the partition act as the equivalence classes.

Example. Let $x \sim y$ iff $x \bmod 2 = y \bmod 2$ over **Z**. Then ~ is an RST with equivalence classes [0], the evens, and [1], the odds. Also {[0], [1]} is a partition of **Z**.

Example. **R** can be partitioned into the set of half-open intervals $\{(n, n + 1] \mid n \in \mathbf{Z}\}$. Then we have an RST ~ over **R**, where $x \sim y$ iff $x, y \in (n, n + 1]$ for some $n \in \mathbf{Z}$.

Quiz (1 minute). In the preceding example, what is another way to say $x \sim y$?

Answer. $x \sim y$ iff $\lceil x \rceil = \lceil y \rceil$.

Refinements of Partitions. If P and Q are partitions of a set S, then P is a *refinement* of Q if every $A \in P$ is a subset of some $B \in Q$.

Example. Let $S = \{a, b, c, d, e\}$ and consider the following four partitions of S.

$P_1 = \{\{a, b, c, d, e\}\}$,
$P_2 = \{\{a, b\}, \{c, d, e\}\}$,
$P_3 = \{\{a\}, \{b\}, \{c\}, \{d, e\}\}$,
$P_4 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$.

Each P_i is a refinement of P_{i-1}. P_1 is the "coarsest" and P_4 is the "finest".

Example. Let $\sim_3$ and $\sim_6$ be the following RSTs over $\mathbf{N}$.

- $x \sim_3 y$ iff $x \bmod 3 = y \bmod 3$ has the following three equivalence classes.

  $[0]_3 = \{3k \mid k \in \mathbf{N}\}$, $[1]_3 = \{3k + 1 \mid k \in \mathbf{N}\}$, $[2]_3 = \{3k + 2 \mid k \in \mathbf{N}\}$.

- $x \sim_6$ y iff $x \bmod 6 = y \bmod 6$ has the following six equivalence classes.

  $[n]_6 = \{6k + n \mid k \in \mathbf{N}\}$ for $n \in \{0, 1, 2, 3, 4, 5\}$.

Notice that $[0]_6 \subset [0]_3$, $[1]_6 \subset [1]_3$, $[2]_6 \subset [2]_3$, $[3]_6 \subset [0]_3$, $[4]_6 \subset [1]_3$, $[5]_6 \subset [2]_3$. So the partition for $\sim_6$ is a refinement of the partition for $\sim_3$.

Quiz (2 minutes). Are either of the RSTs $\sim_3$ and $\sim_2$ refinements of the other?

Answer. No, since $[0]_2$ and $[1]_2$ are the even and odd natural numbers, there is no subset relationship with $[0]_3$, $[1]_3$, and $[2]_3$.

Theorem (*Intersection Property of RST*) If E and F are RSTs over A, then the equivalence classes of $E \cap F$ have the form $[x]_{E \cap F} = [x]_E \cap [x]_F$, where $x \in A$.

Example. Let $\sim_1$ and $\sim_2$ be the following RSTs over **N**.

- $x \sim_1 y$ iff $\lfloor x/4 \rfloor = \lfloor y/4 \rfloor$ has equivalence classes $[4n]_1 = \{4n, 4n + 1, 4n + 2, 4n + 3\}$.
- $x \sim_2 y$ iff $\lfloor x/6 \rfloor = \lfloor y/6 \rfloor$ has equivalence classes $[6n]_2 = \{6n, 6n + 1, \ldots, 6n + 5\}$.

Let $\sim = \sim_1 \cap \sim_2$. Then a few equivalence classes for $\sim$ are:

$[0]_\sim = [0]_1 \cap [0]_2 = \{0, 1, 2, 3\} \cap \{0, 1, 2, 3, 4, 5\} = \{0, 1, 2, 3\}$.

$[4]_\sim = [4]_1 \cap [4]_2 = \{4, 5, 6, 7\} \cap \{0, 1, 2, 3, 4, 5\} = \{4, 5\}$.

$[6]_\sim = [6]_1 \cap [6]_2 = \{4, 5, 6, 7\} \cap \{6, 7, 8, 9, 10, 11\} = \{6, 7\}$.

$[8]_\sim = [8]_1 \cap [8]_2 = \{8, 9, 10, 11\} \cap \{6, 7, 8, 9, 10, 11\} = \{8, 9, 10, 11\}$.

Quiz (2 minutes). Do you see a pattern for the equivalence classes?
Answer.

$[12n]_\sim = \{12n, 12n + 1, 12n + 2, 12n + 3\}$.

$[12n + 4]_\sim = \{12n + 4, 12n + 5\}$.

$[12n + 6]_\sim = \{12n + 6, 12n + 7\}$.

$[12n + 8]_\sim = \{12n + 8, 12n + 9, 12n + 10, 12n + 11\}$.

Generating Equivalence Relations

The smallest equivalence relation containing a binary relation R (i.e., the equivalence closure of R) is $\text{tsr}(R)$.

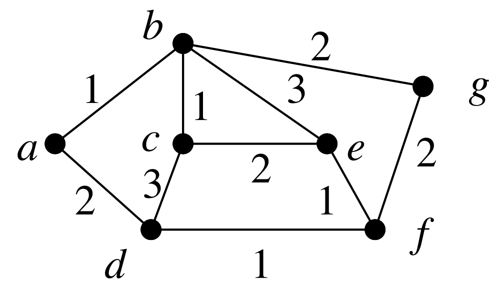
Example. The order *tsr* is important. For example, let $R = \{(a, b), (a, c)\}$ over $\{a, b, c\}$. Then notice that $\text{tsr}(R) = \{a, b, c\} \times \{a, b, c\}$, which is an equivalence relation. But $\text{str}(R) = \{a, b, c\} \times \{a, b, c\} - \{(b, c), (c, b)\}$, which is not transitive.

Kruskal's Algorithm (*minimal spanning tree*)

The algorithm starts with the finest partition of the vertex set and ends with the coarsest partition, where $x \sim y$ iff there is a path between x and y in the current spanning tree.

1. Order the edges by weight into a list L; Set the minimal spanning tree $T := \emptyset$; and construct the initial classes of the form $[v] = \{v\}$ for each vertex v.
2. **while** there are two or more equivalence classes **do**

   $\{x, y\}$:= head(L);
   L := tail(L);
   if $[x] \neq [y]$ **then**
   $T := T \cup \{\{x, y\}\}$;
   replace $[x]$ and $[y]$ by $[x] \cup [y]$
   fi od



Example. For the pictured graph, Step 1 gives a list of edges ordered by weight.

$L := \langle \{a, b\}, \{b, c\}, \{d, f\}, \{e, f\}, \{a, d\}, \{c, e\}, \{f, g\}, \{b, g\}, \{c, d\}, \{b, e\} \rangle$.

T	Equivalence Classes
$\{\ \}$	$\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}$
$T \cup \{\{a, b\}\}$	$\{a, b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}$
$T \cup \{\{b, c\}\}$	$\{a, b, c\}, \{d\}, \{e\}, \{f\}, \{g\}$
$T \cup \{\{d, f\}\}$	$\{a, b, c\}, \{d, f\}, \{e\}, \{g\}$
$T \cup \{\{e, f\}\}$	$\{a, b, c\}, \{d, e, f\}, \{g\}$
$T \cup \{\{a, d\}\}$	$\{a, b, c, d, e, f\}, \{g\}$
Bypass $\{c, e\}$	
$T \cup \{\{f, g\}\}$	$\{a, b, c, d, e, f, g\}$