Section 4.4 Inductive Proof

What do we believe about nonempty subsets of **N**? Since $\langle$ **N**, < $\rangle$ is well-founded, and in fact it is linear, it follows that every nonempty subset has a least element. This little observation is the only thing we need to present the ideas of inductive proof.

Basis for the Principle of Mathematical Induction (PMI)

Let S ⊂ **N**, 0 ∈ S, and suppose also that k ∈ S implies k + 1 ∈ S. Then S = N.

Proof: Assume, BWOC, that S ≠ **N**. Then **N** – S has a least element x. Since 0 ∈ S and x ∉ S, it follows that x > 0. Since x is the least element of **N** – S, it follows that x – 1 ∈ S. So the hypothesis tells us that (x – 1) + 1 ∈ S. So we have x ∉ S and x ∈ S, a contradiction.

Therefore S = **N**. QED.

Principle of Mathematical Induction (PMI)

Let P(n) denote a statement for each n ∈ **N**. To prove P(n) is true for each n ∈ **N** do the following steps.

- Show P(0) is true.
- Show that if P(k) is true then P(k + 1) is true.

Proof: Assume that the two steps have been shown. Let S = {n | P(n) is true}. Since P(0) is true, it follows that 0 ∈ S. Since P(k) true implies P(k + 1) true, it follows that k ∈ S implies k + 1 ∈ S. So by the Basis for PMI we have S = N. QED.

Remark: PMI also works for statements P(n) where n ∈ {m, m + 1, …}. In this case, the least element is m, so step 1 is modified to show that P(m) is true.

Example: Prove that $1 + 2 + \ldots + n = n(n + 1)/2$ for all $n \in \mathbf{N}$.

Proof: Let P(n) be the given equation. For n = 0 the equation is $0 = 0(0 + 1)/2$. So P(0) is true. Now assume P(k) is true and prove P(k + 1) is true. The left side of P(k + 1) is:

$1 + 2 + \ldots + k + (k + 1)$	$= (1 + 2 + \ldots + k) + (k + 1)$	
	$= k(k + 1)/2 + (k + 1)$	(induction assumption)
	$= (k + 1)((k + 1) + 1)/2$	(algebra)

which is the right side of P(k + 1). So P(k + 1) is true and it follows from PMI that P(n) is true for all $n \in \mathbf{N}$. QED.

Example: Prove that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$ for all $n \in \mathbf{N}$.

Proof: Let P(n) be the given equation. For n = 0 the equation is $0^3 = 0^2$. So P(0) is true. Now assume P(k) is true and prove P(k + 1) is true. The left side of P(k + 1) is:

$1^3 + 2^3 + \ldots + k^3 + (k + 1)^3$	$= (1^3 + 2^3 + \ldots + k^3) + (k + 1)^3$	
	$= (1 + 2 + \ldots + k)^2 + (k + 1)^3$	(induction assumption)
	$= (k(k + 1)/2)^2 + (k + 1)^3$	(previous example)
	$= (k^2 + 4k + 4)(k + 1)^2/4$	(algebra)
	$= ((k + 1)(k + 2)/2)^2$	(algebra)
	$= (1 + 2 + \ldots + (k + 1))^2$	(previous example)

which is the right side of P(k + 1). So P(k + 1) is true and it follows from PMI that P(n) is true for all $n \in \mathbf{N}$. QED.

Example: Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) =$ if $n = 0$ then 0 else $f(n - 1) + n^2$.

Claim: $f(n) = n(n + 1)(2n + 1)/6$ for all $n \in \mathbf{N}$.

Proof: Let P(n) be the given equation. Since $f(0) = 0 = 0(0 + 1)(2 \cdot 0 + 1)/6$, P(0) is true. Now assume P(k) is true and prove P(k + 1) is true. The left side of P(k + 1) is:

$f(k + 1)$	$= f(k + 1 - 1) + (k + 1)^2$	(definition of f)
	$= f(k) + (k + 1)^2$	(algebra)
	$= k(k + 1)(2k + 1)/6 + (k + 1)^2$	(induction assumption)
	$= (k + 1)(2k^2 + 7k + 6)/6$	(algebra)
	$= (k + 1)(k + 2)(2k + 3)/6$	(algebra)
	$= (k + 1)((k + 1) + 1)(2(k + 1) + 1)/6$	(algebra)

which is the right side of P(k + 1). So P(k + 1) is true and it follows from PMI that P(n) is true for all $n \in \mathbf{N}$. QED.

Extending Inductive Proof to Well-Founded Sets

A more general form of inductive proof is based on the following statements about well-founded sets.

Basis for Well-Founded Induction

Let W be a well-founded set. Let S be a subset of W that contains the minimal elements of W and whenever an element $x \in W$ has the property that all its predecessors are in S, then $x \in S$. Then S = W.

Proof: Assume, BWOC, that S ≠ W. Then W – S has a minimal element x. So any predecessor of x must be in S. But the assumption then tells us that $x \in S$. So we have $x \notin S$ and $x \in S$, a contradiction. Therefore S = W. QED.

Well-Founded Induction

Let W be a well-founded set. Let P(x) denote a statement for each $x \in W$. To prove P(x) is true for each $x \in W$ do the following steps.

- Show P(m) is true for each minimal element m of W.
- Show that if $x \in W$ and P(y) is true for all predecessors y of x, then P(x) is true.

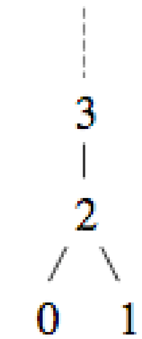
Proof: Assume that the two steps have been shown. Let $S = \{x \in W \mid P(x) \text{ is true}\}$. Since P(m) is true for the minimal elements of W, it follows that S contains the minimal elements of W. The second step tells us that if $x \in W$ and P(y) is true for each $y < x$, then P(x) true. In other words, $x \in W$ and each predecessor of x is in S, then $x \in S$. So by the Basis for Well-founded induction we have S = W. QED.

Example (Well-Founded Induction): Let $f : \mathbf{N} \to \mathbf{N}$ be defined by

$f(0) = f(1) = 0$

$f(n) = f(n - 2) + 1$ (for $n > 1$).

Prove that $f(n) = \lfloor n/2 \rfloor$ for all $n \in \mathbf{N}$



Proof. In this case, we can use a well-founded ordering of **N** that has 0 and 1 as the two minimal elements and with the usual ordering for every other number in **N**. The poset diagram at right indicates this ordering.

Let P(n) be the statement "$f(n) = \lfloor n/2 \rfloor$". The definition of f tells us that $f(0) = f(1) = 0$ and the definition of floor tells us that $\lfloor 0/2 \rfloor = \lfloor 1/2 \rfloor = 0$. So $f(0) = \lfloor 0/2 \rfloor$ and $f(1) = \lfloor 1/2 \rfloor$, which tells us that P(0) and P(1) are true. This takes care of the minimal elements. Now assume that $k > 1$ and P(i) is true (i.e., $f(i) = \lfloor i/2 \rfloor$) for each predecessor $i < k$. We must show that P(k) is true (i.e., $f(k) = \lfloor k/2 \rfloor$). We have the following equations:

$f(k)$	$= f(k - 2) + 1$	(definition of f)
	$= \lfloor (k - 2)/2 \rfloor + 1$	(induction assumption since $k - 2 < k$)
	$= \lfloor (k/2) - 1 \rfloor + 1$	(algebra)
	$= \lfloor (k/2) \rfloor - 1 + 1$	(property of floor)
	$= \lfloor (k/2) \rfloor$	

So P(k) is true and it follows by well-founded induction that P(n) is true for all $n \in \mathbf{N}$.

QED.

A corollary to well-founded induction is the second principle of mathematical induction:

Second PMI

Let P(n) denote a statement for each $n \in \mathbf{N}$. To prove P(n) is true for each $n \in \mathbf{N}$ do the following steps.

- Show P(0) is true.
- Show that if $k > 0$ and P(i) is true for $i < k$ then P(k) is true.

Remark: Second PMI also works for statements P(n) where $n \in \{m, m + 1, \ldots\}$. In this case, the least element is m, so step 1 is modified to show that P(m) is true and step 2 is modified to assume $k > m$.

Proof: **N** is well-founded with minimal element 0. So the result follows from the well-founded induction theorem. QED.

Example: Prove that every natural number greater than 1 is prime or a sum of primes.

Proof: Let P(n) mean that n is a prime or a sum of primes. We need to show P(n) is true for every $n \geq 2$. Since 2 is prime, P(2) is true. Assume $k > 2$ and P(i) is true for $i < k$. Show P(k) is true. If k is prime, then P(k) is true. Otherwise, $k = ij$ where

$$2 \leq i < k \text{ and } 2 \leq j < k$$

So by assumption P(i) and P(j) are true. i.e., i and j are primes or sums of primes. But we can write $k = ij = j + j + \ldots + j$ (i times), so k is a sum of primes. Thus P(k) is true and it follows from Second PMI that P(n) is true for all $n \geq 2$. QED.