Section 5.2 Finding Closed Forms

A *closed form* is an expression that can be computed by applying a fixed number of familiar operations to the arguments. For example, the expression 2 + 4 + … + 2*n* is not a closed form, but the expression *n*(*n*+1) is a closed form.

Summation Notation: $\sum_{i=1}^{n} a_i = a_1 + \cdots + a_n.$

Summation Facts

(1) $\sum c a_i = c \sum a_i.$

(2) $\sum (a_i + b_i) = \sum a_i + \sum b_i.$

(3) $\sum a_i x^{i+k} = x^k \sum a_i x^i.$

(4) $\sum_{i=m}^{n} a_{i+k} = \sum_{i=m+k}^{n+k} a_i.$

The objective is to find closed forms for finite sums.

Some Useful Closed Forms

(1) $\sum_{i=m}^{n} c = (n - m + 1)c.$

(2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$

(3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$

(4) $\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$ (where $a \neq 1$).

(5) $\sum_{i=1}^{n} i a^i = \frac{a - (n+1)a^{n+1} + n a^{n+2}}{(a-1)^2}$ (where $a \neq 1$).

Example. Find a closed form for the expression $\sum_{i=2}^{n}(i-1)2^{i+1}$.

Solution:
$$\sum_{i=2}^{n}(i-1)2^{i+1} = \sum_{i=1}^{n-1} i2^{i+2} \qquad \text{(Fact 4)}$$
$$= 2^2 \sum_{i=1}^{n-1} i2^{i} \qquad \text{(Fact 3)}$$
$$= 2^2(2 - n2^n + (n-1)2^{n+1}) \qquad \text{(Form 5)}$$
$$= 2^3 - (2-n)2^{n+2}.$$

Example. Find a closed form for $2 + 2^2 \cdot 7 + 2^3 \cdot 14 + \cdots + 2^n(n-1) \cdot 7$.

Solution: The sum has the form $2 + \sum_{i=2}^{n} 2^i(i-1) \cdot 7$
$$= 2 + 7\sum_{i=2}^{n}(i-1)2^{i} \qquad \text{(Fact 1)}$$
$$= 2 + 7\sum_{i=1}^{n-1} i2^{i+1} \qquad \text{(Fact 4)}$$
$$= 2 + 14\sum_{i=1}^{n-1} i2^{i} \qquad \text{(Fact 3)}$$
$$= 2 + 14(2 - n2^n + (n-1)2^{n+1}). \qquad \text{(Form 5)}$$

Quiz. Use summation facts and forms to prove that $2+3+\ldots+n=\dfrac{(n-1)(n+2)}{2}$.

Solution.
$$2+3+\ldots+n=\sum_{i=2}^{n} i=\sum_{i=1}^{n-1}(i+1)$$
$$=\sum_{i=1}^{n-1} i+\sum_{i=1}^{n-1} 1$$
$$=\frac{(n-1)(n)}{2}+(n-1)$$
$$=\frac{(n-1)(n+2)}{2}.$$

Quiz. Use summation facts and forms to find a closed form for $3+7+\cdots+(3+4n)$.

Solution.
$$\sum_{i=0}^{n}(3+4i)=\sum_{i=0}^{n} 3+\sum_{i=0}^{n} 4i$$
$$=\sum_{i=0}^{n} 3+4\sum_{i=0}^{n} i$$
$$=3(n+1)+\frac{4n(n+1)}{2}$$
$$=(3+2n)(n+1).$$

Example. Let count(n) be the number of := statements executed by the following algorithm as a function of n, where $n \in \mathbf{N}$. Find a closed form for count(n).

```
i := 1;                              (1)
while i < n do
   i := i + 1;                       (n – 1)
   for j := 1 to i do S od           (2 + 3 + … + n)
od
```

The expressions in parentheses indicate the number of times that := is executed.

Therefore, count(n) is the sum:

$$\begin{aligned} \text{count}(n) &= 1 + (n-1) + (2 + 3 + \cdots + n) \\ &= (n-1) + (1 + 2 + 3 + \cdots + n) \\ &= (n-1) + \frac{n(n+1)}{2}. \end{aligned}$$

Quiz. Let count(n) be the number of executions of S in the preceding algorithm as a function of n. Find a closed form for count(n).

Solution.

$$\begin{aligned} \text{count}(n) &= (2 + 3 + \cdots + n) \\ &= (1 + 2 + 3 + \cdots + n) - 1 \\ &= \frac{n(n+1)}{2} - 1. \end{aligned}$$

Example. Let count(n) be the number of times S is executed by the following algorithm as a function of n, where $n \in \mathbf{N}$. Find a closed form for count(n).

```
i := 1;
while i < n do
   i := i + 2;
   for j := 1 to i do S od
od
```

Solution: Each time through the while-loop i is incremented by 2. So the values of i at the start of each for-loop are 3, 5, …, ($2k + 1$), where $i = 2k + 1 \geq n$ represents the stopping point for the while-loop. So we have

$$\begin{aligned}\text{count}(n) &= 3 + 5 + \cdots + (2k+1) \\ &= \sum_{i=1}^{k}(2i+1) = 2\sum_{i=1}^{k} i + \sum_{i=1}^{k} 1 \\ &= \frac{2k(k+1)}{2} + k = k(k+1) + k = k(k+2).\end{aligned}$$

But we need to write count(n) in terms of n. Since $2k + 1 \geq n$ is the stopping point for the while-loop it follows that $2k - 1 < n$ is the last time the while-condition is true. In other words, we have the inequality $2k - 1 < n \leq 2k + 1$. Solving for k, we have $2k - 2 < n - 1 \leq 2k$, which gives $k - 1 < (n - 1)/2 \leq k$. Therefore $k = \lceil (n - 1)/2 \rceil$. Now we can write count(n) in terms of n as

$$\text{count}(n) = k(k + 2) = \lceil (n - 1)/2 \rceil (\lceil (n - 1)/2 \rceil + 2).$$