Section 6.3 Formal Reasoning

To show that a wff is a tautology we can check its truth table or find an equivalence proof or use a combination of the two (e.g., Quine's method). Another way is formal reasoning.

A *formal reasoning system* consists of three parts: (1) wffs, (2) axioms (a selected subset of wffs assumed to be true), and (3) inference rules (mappings of wffs).

Some Useful Inference Rules

MP (modus ponens-mode that affirms)

$$\frac{A \to B,\ \ A}{\therefore\ \ B}$$

MT (modus tollens-mode that denies)

$$\frac{A \to B,\ \ \neg B}{\therefore\ \ \neg A}$$

Conj (conjunction)

$$\frac{A,\ \ B}{\therefore\ \ A \wedge B}$$

Simp (simplification)

$$\frac{A \wedge B}{\therefore\ \ A}$$

Add (addition)

$$\frac{A}{\therefore\ \ A \vee B}$$

DS (disjunctive syllogism)

$$\frac{A \vee B,\ \ \neg A}{\therefore\ \ B}$$

HS (hypothetical syllogism)

$$\frac{A \to B,\ \ B \to C}{\therefore\ \ A \to C}$$

CD (constructive dilemma)

$$\frac{A \vee B,\ \ A \to C,\ \ B \to D}{\therefore\ \ C \vee D}$$

DD (destructive dilemma)

$$\frac{A \to C,\ \ B \to D,\ \ \neg C \vee \neg D}{\therefore\ \ \neg A \vee \neg B}$$

Proofs

A *proof* is a finite sequence of wffs $W_1, W_2, \ldots, W_n$, where each W_i is either an axiom or in inferred from previous wffs in the sequence. The last wff W_n is a *theorem*.

A *conditional proof* (or *deduction* or *proof from premises*) of B is a finite sequence of wffs ending in B, where each wff is either a premise, an axiom, or is inferred from previous wffs in the sequence. A conditional proof with no premises is a proof.

Conditional Proof Rule (CP) (The Deduction Theorem) *(Its proof comes later)*

For any proof of B from premises $A_1, A_2, \ldots, A_n$ there is a proof of $A_1 \wedge \ldots \wedge A_n \to B$.

Proof Notation. Put each wff on a numbered line along with a reason. Follow the proof with QED. For conditional proofs use the letter P for a premise and write QED followed by the line numbers of the premises, the last line number, and CP under the last line of the proof.

Example. (The simplest conditional proof) The wff $A \to A$ has the following proof.

Proof: 1. A $\quad P$

QED $\quad$ 1, 1, CP.

Example. Prove $(A \vee C \to D) \wedge \neg B \wedge (A \vee B) \to D$.

Proof:

1.	$A \vee C \to D$	P
2.	$\neg B$	P
3.	$A \vee B$	P
4.	A	2, 3, DS
5.	$A \vee C$	4, Add
6.	D	1, 5, MP
	QED	1, 2, 3, 6, CP.

Example. Prove $(A \vee B \to C \wedge D) \wedge A \wedge (C \to E) \to D \wedge E$.

Proof:	1.	$A \vee B \to C \wedge D$	P
	2.	A	P
	3.	$C \to E$	P
	4.	$A \vee B$	2, Add
	5.	$C \wedge D$	1, 4, MP
	6.	C	5, Simp
	7.	E	3, 6, MP
	8.	D	5, Simp
	9.	$D \wedge E$	7, 8, Conj
		QED	1, 2, 3, 9, CP.

Subproofs

A *subproof* is a proof that is part of another proof.

If a subproof is conditional, then indent the statements of the subproof. Also, write down the conditional statement without indentation after the last line of the subproof.

Example. Prove $(A \vee B \to C) \to (A \to C) \vee (B \to C)$.

Proof:	1.	$A \vee B \to C$	P
	2.	A	P
	3.	$A \vee B$	2, Add
	4.	C	1, 3, MP
	5.	$A \to C$	2, 4, CP
	6.	$(A \to C) \vee (B \to C)$	5, Add
		QED	1, 6, CP.

Example/Quiz. Prove $(A \lor B) \to (\neg B \to A)$.

Proof:

1.	$A \lor B$	P
2.	$\neg B$	P
3.	A	1, 2, DS
4.	$\neg B \to A$	2, 3, CP
	QED	1, 4, CP.

Formalizing an Argument

We can often represent an argument by letting each sentence be a wff. Then we can try to assign a reason to each wff and analyze the result to see whether it is a proof.

Quiz(3 minutes). Formalize and prove the following argument.

I eat spinach (S) or ice cream (I). If I study logic (L) then I will pass the exam (P). If I eat ice cream then I will study logic. If I eat spinach then I will play golf (G). I failed the exam. Therefore, I played golf.

Proof:

1.	$S \lor I$	P
2.	$L \to P$	P
3.	$I \to L$	P
4.	$S \to G$	P
5.	$\neg P$	P
6.	$\neg L$	2, 5, MT
7.	$\neg I$	3, 6, MT
8.	S	1, 7, DS
9.	G	4, 8, MP
	QED	1, 2, 3, 4, 5, 9, CP.

Alternative Proof:

1.	$S \lor I$	P
2.	$L \to P$	P
3.	$I \to L$	P
4.	$S \to G$	P
5.	$\neg P$	P
6.	$G \lor L$	1, 3, 4, CD
7.	$\neg L$	2, 5, MT
8.	G	6, 7, DS
	QED	1, 2, 3, 4, 5, 8, CP.

Using Previous Results (T)

We often use one theorem to prove another theorem. When doing so we can treat the known theorem as an axiom. For example, if $X \to Y$ is a known theorem and X is on some line of a proof, then we can write $X \to Y$ on another line with T (for theorem) as the reason. Then MP can be used to write Y. We can simplify the procedure further by omitting the theorem from the proof. In this case, use the reason T to write Y.

Example. Prove $(A \to C) \wedge \neg (A \to B) \to \neg (C \to B)$.

Proof:

1.	$A \to C$	P	
2.	$\neg (A \to B)$	P	
3.	$A \wedge \neg B$	2, T	
4.	A	3, Simp	
5.	C	1, 4, MP	
6.	$\neg B$	3, Simp	
7.	$C \wedge \neg B$	5, 6, Conj	
8.	$\neg (C \to B)$	7, T	{The theorem is $C \wedge \neg B \equiv \neg (C \to B)$.}
	QED	1, 2, 8, CP.	

Example. Prove $(A \to B) \wedge (A \vee B) \to B$.

Proof:

1.	$A \to B$	P	
2.	$A \vee B$	P	
3.	$\quad\quad B$	P	
4.	$B \to B$	3, 3, CP	
5.	$B \vee B$	1, 2, 4, CD	
6.	B	5, T	{The theorem is $B \vee B \equiv B$.}
	QED	1, 2, 6, CP.	

Indirect Proof Rule (IP): If W is a wff, then $W \equiv \neg W \rightarrow$ false. So to prove W, construct a proof of $\neg W \rightarrow$ false. In particular, to prove $X \rightarrow Y$, construct a proof of $X \wedge \neg Y \rightarrow$ false.

Notations: Write "P for IP" as the reason for $\neg W$ (for $\neg Y$ if proving $X \wedge \neg Y \rightarrow$ false). The goal is to find a statement of the form $U \wedge \neg U$, which, by T, is equivalent to false. Write QED followed by the line numbers of the premises, the last line number, and IP under the last line of the proof.

Example (*Disjunctive Syllogism*). Prove $(A \vee B) \wedge \neg B \rightarrow A$.

Proof:	1.	$A \vee B$	P
	2.	$\neg B$	P
	3.	$\neg A$	P for IP
	4.	$\neg A \wedge \neg B$	2, 3, Conj
	5.	$\neg (A \vee B)$	4, T
	6.	$(A \vee B) \wedge \neg (A \vee B)$	1, 5, Conj
	7.	false	6, T
		QED	1, 2, 3, 7, IP.

Example/*Quiz* (*Modus Tollens*). Prove $(A \rightarrow B) \wedge \neg B \rightarrow \neg A$.

Proof:	1.	$A \rightarrow B$	P
	2.	$\neg B$	P
	3.	A	P for IP (use $A \equiv \neg \neg A$ without comment)
	4.	B	1, 3, MP
	5.	$B \wedge \neg B$	2, 4, Conj
	6.	false	5, T
		QED	1, 2, 3, 6, IP.

Example(Absorption). Prove $A \lor (A \land B) \equiv A$ by proving both $A \lor (A \land B) \to A$ and $A \to A \lor (A \land B)$.

Proof of $A \lor (A \land B) \to A$:

1.	$A \lor (A \land B)$	P
2.	$\neg A$	P for IP
3.	$A \land B$	1, 2, DS
4.	A	3, Simp
5.	$A \land \neg A$	2, 4, Conj
6.	false	5, T
	QED	1, 2, 6, IP.

Proof of $A \to A \lor (A \land B)$:

1.	A	P
2.	$A \lor (A \land B)$	1, Add
	QED	1, 2, CP.

Example/Quiz. Use IP to prove $(A \to B) \land (A \lor B) \to B$. Use T only for false.

Proof:

1.	$A \to B$	P
2.	$A \lor B$	P
3.	$\neg B$	P for IP
4.	A	2, 3, DS
5.	B	1, 4, MP
6.	$B \land \neg B$	3, 5, Conj
7.	false	6, T
	QED	1, 2, 3, 7, IP.

Quiz (3 minutes). Use IP somewhere in a proof of $A \to (B \to A)$.

Proof:	1.	A		P
	2.		B	P
	3.		$\neg A$	P for IP
	4.		$A \wedge \neg A$	1, 3, Conj
	5.		false	4, T
	6.	$B \to$ A		2, 3, 5, IP
		QED		1, 6, CP.

Quiz (3 minutes). Find a conditional proof (with no T's) of $A \to (B \to A)$.

Proof:	1.	A		P
	2.		B	P
	3.		$A \vee \neg B$	1, Add
	4.		A	2, 3, DS
	5.	$B \to$ A		2, 4, CP
		QED		1, 5, CP.

Example/Quiz. Find a conditional proof of DD that uses CD.

Proof:	1.	$A \to C$	P
	2.	$B \to D$	P
	3.	$\neg C \vee \neg D$	P
	4.	$\quad\quad \neg C$	P
	5.	$\quad\quad \neg A$	1, 4, MT
	6.	$\neg C \to \neg A$	4, 5, CP
	7.	$\quad\quad \neg D$	P
	8.	$\quad\quad \neg B$	2, 7, MT
	9.	$\neg D \to \neg B$	7, 8, CP
	10.	$\neg A \vee \neg B$	3, 6, 9, CD
		QED	1, 2, 3, 10, CP.

Quiz (3 minutes). Find a conditional proof of $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

Proof:	1.	$A \to (B \to C)$	P
	2.	$\quad A \to B$	P
	3.	$\quad\quad A$	P
	4.	$\quad\quad B$	2, 3, MP
	5.	$\quad\quad B \to C$	1, 3, MP
	6.	$\quad\quad C$	4, 5, MP
	7.	$\quad A \to C$	3, 6, CP
	8.	$(A \to B) \to (A \to C)$	2, 7, CP
		QED	1, 8, CP.